Voronoï cell-size distribution and Edwards' compactivity of the parking lot model

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We find by Monte Carlo simulations that the distribution of Voronoï cell sizes for the parking lot model follows a gamma distribution with shape parameter $k \approx 2$ for high enough packing fractions ϕ. A gamma distribution of Voronoï cells sizes was found recently by Aste *et al.* [Europhys. Lett. **79**, 24003 (2007)] in experiments of static packings of monodisperse spheres. This statistic implies that, for high ϕ, Edwards' compactivity of the parking lot model depends linearly on the average volume per cell, as predicted by the statistical mechanics calculation of Tarjus and Viot, which explicitly counted the *blocked* configurations of this model [Phys. Rev. E **69**, 011307 (2004)].

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Granular materials are athermal systems composed of a large number of particles with nonlinear and dissipative interactions. Predicting the collective behavior of these systems is relevant in numerous human activities [1]. A statistical mechanics description of granular materials in mechanical equilibrium was proposed by Edwards and coauthors in 1989 in which the total volume V_T of the system plays the usual role of energy in conventional statistical mechanics [2]; a flat measure over all the *blocked* states for a given volume is assumed. In this description, the equivalent of the thermodynamic temperature is the compactivity $\chi = (\partial S/\partial V_T)^{-1}$, where S is the entropy of the system. In mechanical equilibrium, sphere packings can be found with packing fractions ϕ (fraction of the total volume occupied by the grains) in the range $\phi_{\mathrm{RLP}} - \phi_{\mathrm{RCP}}$. Large compactivities are expected for ϕ near the random loose packing limit ϕ_{RLP} and small ones for packing fractions near the random close packing limit ϕ_{RCP}.

If the total volume of the sample is a state variable, then the distribution of volumes at the grain level is fundamental. In a recent article, Aste *et al.* [3] studied the local volume distribution of monodisperse sphere packings with packing fractions covering most of the range between ϕ_{RLP} and ϕ_{RCP}. Aste *et al.* were able to collapse a large amount of experimental data on a single invariant distribution of local volumes: in three dimensions, the distribution of local Voronoï volumes of monodisperse sphere packings in mechanical equilibrium is a gamma distribution with shape parameter $k \approx 12$. A gamma distribution of shape parameter k is the distribution of the sum of k independent exponentially distributed variables. Simple models that exhibit this local volume distribution could help to understand the origin of the statistical robustness found by Aste *et al.* In this Brief Report, we study the distribution of local Voronoï volumes of the parking lot model [4–10], which is an off-lattice stochastic model of adsorption and desorption of particles on a substrate. We find that this model exhibits a gamma distribution of local Voronoï volumes only for high enough packing fractions. The resulting compactivity of this model is in agreement, in this range of ϕ, with the results of a statistical mechanics calculation made by Tarjus and Viot [10].

Nowak *et al.* [6] introduced the parking lot model (PLM) in the context of granular materials. In this model, particles of unit length adsorb uniformly on a substrate with rate p_+ and desorb with rate p_-; desorption is unrestricted, while a particle can adsorb on the substrate only if it does not overlap with other particles. Once adsorbed, a particle does not move until it desorbs. For a given initial condition of the substrate, the model converges to a stationary state of packing fraction ϕ_e that fluctuates, with ϕ_e depending only on the parameter $K = p_+/p_-$. For large K, ϕ_e is given by $\phi_e \simeq 1 - 1/\log K$ [4]. For $\phi_e > \phi_c$, the PLM exhibits very slow relaxation, reminiscent of a glasslike behavior [7,11]. $\phi_c \approx 0.75$, which corresponds to $K_c \approx 60$. ϕ_c equals the packing fraction at which the irreversible PLM, the case in which $p_- = 0$, jams (starting from an empty substrate) [7]. This one-dimensional model is thought to represent an average column of grains in a granular pack.

From the assumption that space can be divided into elementary cells that can have any volume larger than or equal to a minimum volume, that the sum of the volumes of these elementary cells is the total volume of the sample, that k elementary cells form each cell of a given partition of space, and that any assembly of those cells produces stable packings, Aste *et al.* arrived at the following distribution function [3]:

$$f(V,k) = \frac{1}{\Gamma(k)} \frac{(V - V_{\min})^{(k-1)}}{\chi^k} \exp\left(-\frac{V - V_{\min}}{\chi}\right), \qquad (1)$$

where $f(V,k)$ is the probability of a cell of volume V (that consists of k elementary cells), χ is the compactivity, which is given by

$$\chi = \frac{\langle V \rangle - V_{\min}}{k}, \qquad (2)$$

and $\langle V \rangle$ is the average volume per cell. $f(V,k)$ in Eq. (1) is a gamma distribution, in the variable $V - V_{\min}$, with shape parameter k and scale parameter χ.

Voronoï decomposition is a common and useful way to divide space into pieces. The Voronoï cell of a given particle in the PLM consists of all those points that are closer to that particle than to any other particle on the substrate. In Fig. 1(a), we show the distribution of Voronoï lengths $p(x)$ of the PLM obtained from Monte Carlo simulations for several val-

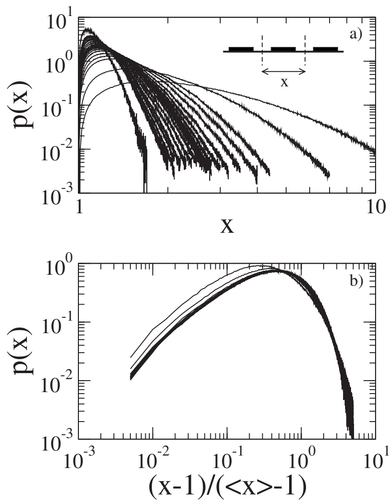

FIG. 1. Voronoï length distribution $p(x)$ of the PLM. (a) $p(x)$ versus x. (b) The same data of (a) but, as suggested by Eq. (1), we plot $p(x)$ versus $(x-x_{\min})/(\langle x\rangle - x_{\min})$. We show, for $L=100$, $p(x)$ for $K=2$, 4, 10, 14, 20, 30, 40, 50, 60, 65, 70, 80, 100, 140, 200, 300, 400, 600, 800, 1000, and 10000. For the PLM, $\phi=1/\langle x\rangle$ and $x_{\min}=1$.

ues of K. As suggested by Eq. (1), we plot $p(x)$ as a function of $(x-x_{\min})/(\langle x\rangle - x_{\min})$ in Fig. 1(b), and a partial collapse for $p(x)$ on a single curve can be observed.

Aste and Di Matteo [12] showed that the parameter k of Eq. (1) can be obtained from $\langle V\rangle$ and the variance σ_V^2,

$$k = \frac{(\langle V\rangle - V_{\min})^2}{\sigma_V^2}, \tag{3}$$

and that k follows the relation

$$k = \frac{\partial \langle V\rangle}{\partial \chi}. \tag{4}$$

From this last equation, the authors identified the parameter k as the analogue of the specific heat [12], so k measured from Eq. (3) must be sensitive to the internal organization of the system. In Fig. 2, we show the parameter k calculated

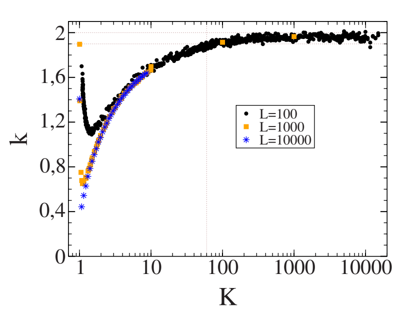

FIG. 2. (Color online) k calculated from Eq. (3) as a function of the parameter K of the PLM for $L=100$, 1000, and 10 000. Each point in this graph corresponds to an average over 1000 samples in stationary state. A finite size *transition* can be observed as the parameter $K\to 1$, manifested as an extremum in $k(K)$ [12].The vertical dotted line corresponds to $K=K_c$.

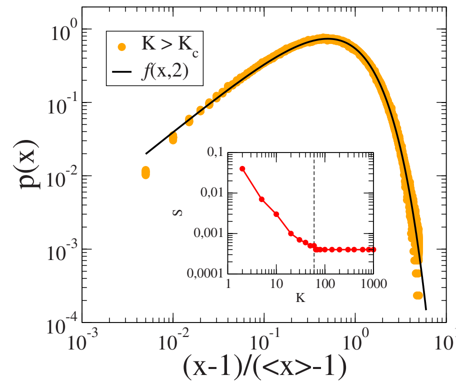

FIG. 3. (Color online) Voronoï length distribution $p(x)$ of the PLM. The same data of Fig. 1(b) are plotted here only for $K>K_c\approx 60$. The continuous line is a gamma distribution with $k=2$. In the inset, we show $S=\Sigma[p(x)-f(x,2)]^2$ vs K for $L=200$. A convergence to $f(x,2)$ can be seen at $K\approx K_c$. In calculating S, $(x-x_{\min})/(\langle x\rangle - x_{\min})$ in the interval 0–5 was divided in 1000 subintervals.

from Eq. (3) for the PLM as a function of K. It can be seen that for $K>K_c$, the PLM saturates to $k\approx 2$.

Not all the curves in Fig. 1 are consistent with Eq. (1). Consider the case $K=2$. From Fig. 2, we read that $k\approx 1$ for $K\approx 2$, but for $k\approx 1$ Eq. (1) reduces to a pure exponential that is inconsistent with the distribution $p(x)$ (the broader one) that can be seen in Fig. 1 for $K=2$. We find that only for $K>K_c$ is the measured $p(x)$ consistent with a gamma distribution. In Fig. 3, we plot $p(x)$ for $K>K_c$, and a very good agreement with a gamma distribution with shape parameter $k=2$ can be observed.

If the distribution of local volumes is given by a gamma distribution like Eq. (1), then the compactivity is given by Eq. (2). For the PLM then, we have that the compactivity is given by

$$\chi_{\mathrm{PLM}} = \frac{1}{k}\frac{1}{\log K - 1} \tag{5}$$

for $K>K_c$.

Tarjus and Viot [10] have worked out a statistical mechanics description of the PLM within the framework proposed by Edwards. As pointed out by these authors, in the PLM there is no explicit account of a mechanical stability condition. They assumed that the stable or blocked configurations in the PLM are those for which no more particle insertions are possible, i.e., all those configurations for which the available line fraction Φ is zero. For a given ϕ (for a given number N of particles), the authors calculated the number of configurations for this model under the constraint that all the N gaps in the substrate have lengths smaller than 1 ($\Phi=0$). Assuming, as proposed by Edwards, that all those configurations are equiprobable and in the limit of a large system and a large value of the parameter K (large packing fractions), Tarjus and Viot (TV) obtained that the compactivity and the packing fraction of the PLM satisfy the relation [10]

$$\frac{1-\phi}{\phi} = \chi_{\mathrm{TV}} - \frac{\exp(-1/\chi_{\mathrm{TV}})}{1-\exp(-1/\chi_{\mathrm{TV}})}. \tag{6}$$

Since for large K small compactivities are expected, Eq. (6) reduces to

$$\frac{1-\phi}{\phi} \approx \chi_{\mathrm{TV}} \tag{7}$$

for large values of K. Since $\phi = 1/\langle x \rangle$, Eq. (7) is consistent with a gamma distribution of shape parameter $k=1$, which is expected since the partition used by the authors consists of gaps with half-particles on each side.

From the results for the PLM presented above, volume exclusion and disorder appear to be crucial ingredients to observe gamma distributions in granular materials. Anyway, it is clear that not all the system's properties are embedded in their distribution of local volumes. The same local volume distribution describes monodisperse sphere packings in almost the whole range of packing fractions from ϕ_{RLP} to ϕ_{RCP}. However, in this range of packing fractions, Schröter *et al.* [13] showed that the system's response to shear exhibits a phase transition at a packing fraction ϕ_m, with $\phi_{\mathrm{RLP}} < \phi_m < \phi_{\mathrm{RCP}}$.

In this Brief Report, we have studied the distribution of local Voronoï lengths of the PLM, and we have found that it exhibits a gamma distribution of shape parameter $k \approx 2$ for high enough packing fractions. This type of distribution of local Voronoï volumes has been found experimentally by Aste *et al.* [3] for monodisperse sphere packings in mechanical equilibrium. As a consequence of this type of distribution, the Edwards compactivity depends linearly on the average volume per cell, a result that is consistent with the statistical mechanics calculation made by Tarjus and Viot [10] for the PLM.

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